

Then $\lambda_n \geq \lambda_{n+1} \geq \dots$, so that
 $\lim_{n \rightarrow \infty} \lambda_n = \lambda \geq 0$.

Claim $\lambda = 0$. Indeed, let $y_n \in \mathcal{H}_n^\perp$,
 $\|y_n\| = 1$ be s.t.

$$\|T y_n\| \geq \frac{\lambda}{2}.$$

Since $y_n \xrightarrow{w} 0$, $T y_n \rightarrow 0$ and $\lambda = 0$.

Now set

$$T_n x = \sum_{i=1}^n (x, x_i) T x_i, \quad \underline{x \perp \text{Ker } T}$$

Using $x = \sum_{i=1}^{\infty} (x, x_i) x_i$,

we get

$$\|(T - T_n) x\| \leq \lambda_n \|x\|.$$

Setting $T_n|_{\text{Ker } T} = 0$, we get
 from here that

$$\lim_{n \rightarrow \infty} \|T - T_n\| = 0.$$

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Thm Let $f: D \rightarrow \mathcal{L}(\mathcal{H})$
 be analytic operator-valued function
 s.t. $f(z)$ is compact for all $z \in D$.

Then either (D - domain)

(a) $(I - f(z))^{-1}$ does not exist $\forall z \in D$;
or

(b) $\exists S \subset D$, discrete, s.t.

$(I - f(z))^{-1}$ exists for all $z \in D \setminus S$.
Then $(I - f(z))^{-1}$ is meromorphic in D ,
analytic in $D \setminus S$, the residues are
finite rank operators and for $z \in S$
 $f(z)x = x$ has nonzero solution in
 \mathcal{H} .

Proof

$\forall z_0 \in D$ let $r > 0$ be s.t. $\{ |z - z_0| < r \} \subset D$
and $\|f(z) - f(z_0)\| < \frac{1}{2}$. Let F be a
finite rank operator² s.t. $\|f(z_0) - F\| < \frac{1}{2}$.
Then for $z \in D_r$

$$\|f(z) - F\| < 1,$$

so that $(I - f(z) + F)^{-1}$ exists and is
analytic in D_r .

Will use the key identity

$$\underbrace{(I - f(z))}_{\text{invertible}} = \underbrace{(I - F(I - f(z) + F)^{-1})}_{\|g(z)\|}$$

$$(I - f(z) + F).$$

Important because $g(z)$ is of finite
rank with the same image (actually
 $= \text{Im } F$) for $z \in D_r$.

F is of finite rank: $\exists y_1, \dots, y_N \in \mathcal{H}$
 s.t.

$$F(x) = \sum_{i=1}^N l_i(x) y_i = \sum_{i=1}^N (x, x_i) y_i$$

by Riesz lemma.

Let $\varphi_n(z) = (\mathbf{I} - f(z) + F^{-1})^* (x_n) \in \mathcal{H}$,
 then

$$g(z) = \sum_{n=1}^N (\cdot, \varphi_n(z)) y_n.$$

Statement for $f(z)$ reduces to the
 statement for $g(z)$, which is linear algebra!
 Namely,

$$g(z)x = x \Leftrightarrow \det \left(\delta_{nm} - (y_n, \varphi_m(z)) \right) = 0$$

analytic in $D_r \setminus d(z)$

$$(x = \sum_{n=1}^N c_n y_n)$$

$$\Rightarrow \det \equiv 0 \text{ in } D_r$$

\det has discrete set of zeros.

In the latter case, if $d(z) \neq 0$, then
 $(\mathbf{I} - f(z))^{-1}$ exists. If $d(\tilde{z}) = 0$, then

$(\mathbf{I} - g(z))^{-1}$ has a pole at $z \rightarrow \tilde{z}$

(using cofactor form of solutions to

$$(\mathbf{I} - g(z))x = y).$$

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$$\text{Let } F = \sum_{i=1}^N (\cdot, x_i) y_i.$$

$$\text{Im}(I - F) = \mathcal{H} \Leftrightarrow$$

$$\det(\delta_{nm} - (y_n, x_m)) \neq 0$$

Indeed, $\forall y \in \mathcal{H}$

$$y = x - Fx = x - \sum_{i=1}^N (x, x_i) y_i$$

$$\text{Set } x = y + \sum_{i=1}^N c_i y_i,$$

$$\sum_{i=1}^N c_i y_i = \sum_{i,j=1}^N c_i (y_i, x_j) y_j$$

$$= \sum_{i=1}^N (y, x_i) y_i, \text{ or}$$

$$c_i - \sum_{j=1}^N a_{ij} c_j = b_i,$$

$$\left. \begin{array}{l} x = y + v \\ (I - F)x = y \\ \Downarrow \\ (I - F)v = Fy \end{array} \right\}$$

where

$$a_{ij} = (y_j, x_i),$$

$$b_i = (y, x_i)$$

$V = \mathbb{C}y_1 \oplus \dots \oplus \mathbb{C}y_N$, $F: V \rightarrow V$
 $(I - F)^{-1} \Leftrightarrow F|_V$ is onto.

Fredholm 1st Theorem

Let $A \in \mathcal{L}(\mathcal{H})$ be compact. Then $\lambda \neq 0$ is an eigenvalue for A iff $\bar{\lambda} \neq 0$ ——— || ——— || ——— A^*

Proof Take $f(z) = zA$, $\lambda = \frac{1}{z}$.

λ is an eigenvalue $\Leftrightarrow I - g(z)$ is not invertible $\Leftrightarrow \det(\delta_{nm} - (y_n, \varphi_m(z))) = 0 \Rightarrow \bar{\lambda}$ is an eigenvalue of A^* , since $I - g(z)^*$ is invertible $\Leftrightarrow \det(\delta_{nm} - (y_n, \varphi_m(z))) \neq 0$.

Fredholm 2nd Theorem

(a) $A - \lambda I$, $\lambda \neq 0$ is invertible iff $\text{Ker}(A - \lambda I) = \{0\}$

(b) $Ax - \lambda x = y$ is solvable iff $y \perp \text{Ker}(A^* - \bar{\lambda} I)$.

Proof Again $f(z) = zA$, $\lambda = \frac{1}{z}$

Part (a) immediate. Part (b):

$$\text{Ker}(A^* - \bar{\lambda} I)^\perp = \underline{\text{Im}(A - \lambda I)}$$

$$-85' - \begin{pmatrix} I & 0 \\ * & T_1 \end{pmatrix}^* = \begin{pmatrix} I & * \\ 0 & T_1^* \end{pmatrix}$$

If

$$g(z)x = \sum_{n=1}^N (x, \varphi_n(z)) y_n,$$

then

$$g(z)^* y = \sum_{n=1}^N (y, y_n) \varphi_n(z)$$

and

$$\frac{\det(\delta_{nm} - (y_n, \varphi_m(z)))}{\det(\delta_{nm} - (\varphi_n(z), y_m))}.$$

This completes 1st theorem.
proof of the

Fact $\dim \mathcal{H} = \infty$, then

$$\sigma(A) \ni 0$$

(if A^{-1} exists, $I = A^{-1}A$ would
 be compact)

$$\text{Im } A \neq \text{Im } A$$

(otherwise

$$A|_{\text{Ker } A^\perp} : \text{Ker } A^\perp \xrightarrow{\sim} \text{Im } A$$

has inverse)

~~.....~~
 $= \text{Im}(A - \lambda I),$

since $A - \lambda I = (\text{I} - \text{finite rank operator}) \times \text{invertible}$

Indeed, if $T = I + F$, then

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_\perp$$

" |
 Ker F finite-dimensional

and in a block form T is

$$\begin{pmatrix} I & 0 \\ * & I + F \end{pmatrix} \approx \begin{pmatrix} I & 0 \\ * & T_\perp \end{pmatrix}$$

Thus $\text{Im } T = \mathcal{H}_0 \oplus \underbrace{\text{Im } T_\perp}_{\text{finite-dim}}$ - closed.

Fredholm 3rd thm For $\lambda \neq 0$,

$$\dim \text{Ker}(A - \lambda I) = \dim \text{Ker}(A^* - \bar{\lambda} I).$$

Proof Follows from $f(z) = zA, z = \frac{1}{\lambda}$, and the same fact for finite-dimensional spaces.

Thm 1 (Riesz-Schauder). Let $A \in \mathcal{L}(\mathcal{H})$ be compact. Then $\sigma(A)$ is a discrete set with only possible accumulation point at 0. Moreover, every non-zero $\lambda \in \sigma(A)$ is an eigen-value of finite

multiplicity.

Proof

Again $f(z) = zA$ - analytic everywhere, so that

$$zAx = x, \quad x \in \mathcal{H}, \quad x \neq 0$$

- discrete set of z . If $\frac{1}{\lambda}$ is not in this set,

$$(\lambda - A)^{-1} = \frac{1}{\lambda} (1 - \frac{1}{\lambda} A)^{-1}.$$

Thm 2 (Hilbert - Schmidt) Let $A = A^*$ be compact on \mathcal{H} . Then $\exists \{x_n\}$ - an orthonormal basis for \mathcal{H} s.t.

$$Ax_n = \lambda_n x_n$$

and $\lim_{n \rightarrow \infty} \lambda_n = 0$ if $\dim \mathcal{H} = \infty$. \implies

Lemma If $A = A^* \neq 0$ and compact on \mathcal{H} , then A has non-zero eigen-value.

Proof

If not, $\sigma(A) = \{0\}$ and then

$$\|A\| = 0, \text{ i.e. } A = 0.$$

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Now then follows

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Thm 3 (Canonical form of compact operator) Let A be compact on \mathcal{H} . Then \exists orthonormal sets $\{x_n\}_{n=1}^{\infty}$ & $\{y_n\}_{n=1}^{\infty}$ and positive $\{\mu_n\}_{n=1}^{\infty}$ (called singular values) s.t.

$$A = 0 \cdot P_0 + \sum_{k=1}^{\infty} \lambda_k P_k$$

- spectral decomposition

$$A = \sum_{n=1}^{\infty} \mu_n (\cdot, x_n) y_n \quad (*)$$

- converges in norm.

Proof Let $\lambda_n = \mu_n^2$ be eigen-values $\neq 0$ for A^*A with eigenvectors x_n .

Set

$$y_n = \frac{1}{\sqrt{\mu_n}} A x_n, \quad (y_n, y_m) = \delta_{nm},$$

then

$$\begin{aligned} Ax &= \sum_{n=1}^{\infty} (Ax, y_n) y_n \\ &= \sum_{n=1}^{\infty} \mu_n (x, x_n) y_n. \end{aligned}$$

since $\{x_n\} = \text{Ker } A^\perp$.

Expansion (*) converges in norm:

$$\left\| \sum_{n=N}^{\infty} \mu_n (x, x_n) y_n \right\|^2$$

$$\leq \max_{n \geq N} \lambda_n \|x\|^2, \text{ and } \lambda_n \rightarrow 0.$$

$$n \geq N$$

When A self-adjoint, (*) becomes

$$A = \sum_{n=1}^{\infty} \mu_n (\cdot, x_n) x_n. \quad |\mu_n| = \lambda_n, \quad \mu_n - e\text{-values of } A,$$

Counter-example to the Lemma when $A \neq A^*$:

$$(Tf)(x) = \int_0^x K(x,t) f(t) dt$$

has non-zero eigenvalues on $L^2([0,1])$.

(Here $K(x,y) \in C([0,1] \times [0,1])$)

Volterra operator

For instance, let $K \equiv 1$:

$$\int_0^x f(t) dt = \lambda f(x),$$

$$f(0) = 0 \text{ \& } \lambda f'(x) = f(x) \Rightarrow f \equiv 0.$$

(In general, prove that for $\lambda \neq 0$

$Tf - \lambda f = g$ is solvable
by iterations)

- 90'' -

Better proof ($\text{Ker } F = \{0\}$).

As before, $\mathcal{H} = \text{Im } F + V$.

Put $\hat{\mathcal{H}} = \mathcal{H} \oplus V$ - H. space,

$$\hat{F}(x, v) = Fx + v.$$

$$\hat{F}: \hat{\mathcal{H}} \rightarrow \mathcal{H} \quad 1-1 \text{ \& onto}$$

$\Rightarrow \hat{F}$ is top. isomorphism. Now

$$\hat{F}^{-1}(\text{Im } F) = \mathcal{H} \oplus \{0\} \subset \hat{\mathcal{H}}$$

-closed, so that $\text{Im } F$ is also closed.

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Remark Condition $\text{Im } F$ is closed follows from $\dim(\mathcal{H}/\text{Im } F)$ is finite-dim as a linear subspace. Indeed, if it is so $\exists V \subset \mathcal{H}$ -subspace, $\dim V < \infty$ s.t.

$$\mathcal{H} = \text{Im } F + V$$

(as linear spaces). Since

$$\mathcal{H} = V^\perp \oplus V,$$

let $P: \mathcal{H} \rightarrow V^\perp$ be orthogonal projection.

Then

$G = PF: \mathcal{H} \rightarrow V^\perp$ is bounded & onto. Assuming $\text{Ker } F = 0$ (replace \mathcal{H} by $\mathcal{H}/\text{Ker } F$), G is linear isomorphism. By open mapping thm, $G^{-1}: V^\perp \rightarrow \mathcal{H}$ is bounded.

Let $y_n = F(x_n) \rightarrow y$.

Then $G(x_n) \rightarrow Py \Rightarrow$

$x_n \rightarrow x = G^{-1}Py$. Thus $y = F(x)$ & $\text{Im } F$ is closed.

III.3. Fredholm operators

Def $F \in \mathcal{L}(\mathcal{H})$ is Fredholm, if $\text{Im } F \subset \mathcal{H}$ is closed and $\dim \text{Ker } F, \dim \text{Ker } F^* < \infty$.

Def $\text{index } F = \dim \text{ker } F - \dim \text{ker } F^*$.

Thm Let $F \in \mathcal{L}(\mathcal{H})$ be Fredholm.

Then

(a) $\exists \varepsilon > 0$ s.t. all $\|G - F\| < \varepsilon$ are Fredholm and $\text{ind } F = \text{ind } G$

(b) If K compact, then $F + K$ Fredholm and $\text{ind}(F + K) = \text{ind } F$.

(c) Operator F is Fredholm iff $\exists G \in \mathcal{L}(\mathcal{H})$ s.t. $K = FG - I$ & $K' = GF - I$ are compact.

(d) If F and G are Fredholm, then FG is also and

$$\text{ind } FG = \text{ind } F + \text{ind } G$$

(e) $\text{ind } F^* = -\text{ind } F$

Examples

(1) $F \in \mathcal{L}(\mathcal{H})$, F is invertible

(2) $F = I + K$, K - compact.

III.4. Schatten ideals \mathcal{S}_1 and \mathcal{S}_2 :

III.4.1. The trace class and Hilbert-Schmidt operators

Def Let $A \geq 0$, $A \in \mathcal{L}(\mathcal{H})$, \mathcal{H} - a separable Hilbert space.

$$\text{tr} A = \sum_{n=1}^{\infty} (A x_n, x_n)$$

for any orthonormal basis of \mathcal{H} .

Thm 1. (a) $\text{tr} A$ is well-defined

$$(b) \text{tr}(\alpha A + \beta B) = \alpha \text{tr} A + \beta \text{tr} B,$$

$$\alpha, \beta \geq 0$$

$$(c) \text{tr} U A U^{-1} = \text{tr} A \text{ for every unitary } U$$

$$(d) 0 \leq A \leq B \Rightarrow 0 \leq \text{tr} A \leq \text{tr} B.$$

Proof Everything follows from property (a). To prove (a), let $\{y_n\}$ be another orthonormal basis for \mathcal{H} . Then

$$\begin{aligned} \sum_{n=1}^{\infty} (A x_n, x_n) &= \sum_{n=1}^{\infty} \|\sqrt{A} x_n\|^2 \\ &= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} |(\sqrt{A} x_n, y_m)|^2 \right) \end{aligned}$$

(change the order of summation)

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |(\sqrt{A} y_m, x_n)|^2$$

$$= \sum_{n=1}^{\infty} \|\sqrt{A} y_n\|^2 = \sum_{n=1}^{\infty} (A y_n, y_n).$$

Def $A \in \mathcal{L}(\mathcal{H})$ is of trace class ($A \in \mathcal{S}_1$) if $\text{tr} |A| < \infty$.

Thm 2 \mathcal{S}_1 is two-sided *-ideal in $\mathcal{L}(\mathcal{H})$.

Proof clearly, $|\lambda A| = |\lambda| |A|$. Next, if $A, B \in \mathcal{S}_1$. Set

$$A + B = U |A + B| \quad (|A| = U^* A, \text{ etc.})$$

$$A = V |A|, B = W |B|. \quad (U^* U = P_{\overline{\text{Im} |A|}})$$

Then

$$\sum_{n=1}^{\infty} (x_n, |A + B| x_n) = \sum_{n=1}^{\infty} (x_n, U^* (A + B) x_n)$$

$$\leq \sum_{n=1}^{\infty} |(x_n, U^* V |A| x_n)| + \sum_{n=1}^{\infty} |(x_n, U^* W |B| x_n)|$$

Consider the first sum; by Cauchy-Schwarz, it is

$$\leq \left(\sum_{n=1}^{\infty} \| |A|^{1/2} V^* U x_n \|^2 \right)^{1/2}$$

$$\left(\sum_{n=1}^{\infty} \| |A|^{1/2} x_n \|^2 \right)^{1/2}$$

$$= \text{tr} |A|.$$

Clearly,

$$\sum_{n=1}^{\infty} \left\| |A|^{1/2} V^* U x_n \right\|^2$$

$$\leq \operatorname{tr} (U^* V |A| V^* U) \leq \operatorname{tr} (V |A| V^*) \leq \operatorname{tr} |A|,$$

so that

$$\operatorname{tr} |A+B| \leq \operatorname{tr} |A| + \operatorname{tr} |B|$$

(not that $|\cdot|$ does not satisfy triangle inequality!) This proves \mathcal{S}_1 is a subspace
Lemma Every $B \in \mathcal{L}(\mathcal{H})$ is a linear combination of four unitary operators.

Proof
$$B = \frac{1}{2} \underbrace{(B+B^*)}_{\text{self-adjoint}} + \frac{i}{2i} \underbrace{(B-B^*)}_{\text{self-adjoint}}$$

If A is self-adjoint, $\|A\| \leq 1$, then $U_{\pm} = A \pm i\sqrt{I-A^2}$ are unitary!

$$A = \frac{U_+ + U_-}{2}$$

Now let $A \in \mathcal{S}_1$, $B \in \mathcal{L}(\mathcal{H})$; using Lemma, we need to verify that

$UA, AV \in \mathcal{S}_1$, where U is unitary.

But $|UA| = |A|,$

$$|AV| = |U^{-1}AU| = U^{-1}|A|U.$$

This proves that \mathcal{S}_1 is a two-sided ideal.

Finally, let $A = U|A|$ and

$A^* = V|A^*|$, then $|A^*| = V^*|A|U^*$ and as we have shown,

$$\text{tr } |A| < \infty \Rightarrow \text{tr } |A^*| < \infty$$

(because of the two-sided ideal property)

Corollary \mathcal{S}_1 is a Banach space with the norm

$$\|A\|_1 = \text{tr } |A|;$$

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$$\|A\| \leq \|A\|_1.$$

Everything is clear except completeness with respect to $\|\cdot\|_1$ norm. Then, $\|x\| = 1,$

$$(|A|x, x) = \sum_{n=1}^{\infty} (|A|x, x_n) (x_n, x) \quad \left(|A|^{1/2} \right) \quad \left(|A|^{1/2} \right)$$

$$\leq \sum_{n=1}^{\infty} \| |A|^{1/2} x_n \|^2 = \text{tr } |A|.$$

$n=1$

Completeness: exercise.

Thm 3 Every $A \in \mathcal{S}_1$ is compact.
 A compact $A \in \mathcal{S}_1$ if and only if

$$\sum_{n=1}^{\infty} \mu_n < \infty.$$

Proof $A \in \mathcal{S}_1$, so $|A|^2 \in \mathcal{S}_1$, so

$$\text{tr } |A|^2 = \sum_{n=1}^{\infty} \|A x_n\|^2 < \infty, \text{ so } \{x_n\} \text{ - O.N.B.}$$

that $\forall x \in \mathcal{H}, \sqrt{\|x\|=1}, x \perp \{x_1, \dots, x_N\}$

$$\|Ax\|^2 \leq \text{tr } |A|^2 - \sum_{i=1}^N \|Ax_i\|^2$$

$$\left\| \left(A - \sum_{i=1}^N (\cdot, x_i) A x_i \right) x \right\|^2 \rightarrow 0 \text{ as } N \rightarrow \infty$$

and if $x \in \{x_1, \dots, x_N\}$, the difference is 0!

This proves that A is compact. By definition,

$$\text{tr } |A| = \sum_{n=1}^{\infty} \mu_n,$$

for the basis consisting of eigen-vectors of $|A|^2$, $\lambda_n = \mu_n^2$. $\|\cdot\|_2$

Corollary Finite rank operators are dense in \mathcal{S}_1 .

Corollary

$$\text{tr } |A| = \text{tr } |A^*|$$

Thm 4 If $A \in \mathcal{S}_1$, then for any orthonormal basis $\{z_n\}$ of \mathcal{H} (and independent of the basis)

$$\sum_{n=1}^{\infty} |(Az_n, z_n)| \leq \sum_{n=1}^{\infty} \mu_n < \infty.$$

Proof Since A is compact, by Hilbert-Schmidt theorem, $\forall x \in \mathcal{H}$

$$x = x_0 + \sum_{n=1}^{\infty} (x, x_n) x_n$$

and

$$Ax = \sum_{n=1}^{\infty} \mu_n (x, x_n) y_n,$$

so that

$$(*) \quad (Az_n, z_n) = \sum_{i=1}^{\infty} \mu_i (z_n, x_i) (y_i, z_n)$$

and

$$\sum_{n=1}^{\infty} |(Az_n, z_n)| \leq \sum_{i=1}^{\infty} \mu_i.$$

$$\begin{aligned} & \left(\sum_{n=1}^{\infty} |(z_n, x_i)|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} |(y_i, z_n)|^2 \right)^{1/2} \\ &= \sum_{i=1}^{\infty} \mu_i. \end{aligned}$$

$\text{Tr } A := \sum_{n=1}^{\infty} (Az_n, z_n)$
 is well-defined

Finally, summing over n in (*) we get

$$\sum_{n=1}^{\infty} (A z_n, z_n) = \sum_{i=1}^{\infty} \mu_i (y_i, x_i),$$

which does not depend on the choice of the basis.

Without proof.

Lidskij thm If $A \in \mathcal{S}_1$ and

$\{\lambda_k\}$ are its eigenvalues, then

$$\text{tr } A := \sum_{n=1}^{\infty} (A x_n, x_n) = \sum_k \lambda_k.$$

(If A has no eigenvalues, $\text{tr } A = 0$).

Gohberg-Krein thm If $A \in \mathcal{L}(\mathcal{H})$

and for some orthonormal basis

$\{x_n\}$ for \mathcal{H}

$$\sum_{n=1}^{\infty} |(A x_n, x_n)| < \infty,$$

then $A \in \mathcal{S}_1$.

(Will prove it later for $A \geq 0$)

In general: $\sum_n (A x_n, x_n) < \infty \forall \{x_n\}$ - basis \mathcal{H} ,
then $A \in \mathcal{S}_1$.

Not quite true: true for $A \geq 0$

III.4.2. The Hilbert-Schmidt class

Def $T \in \mathfrak{S}_2$ if $\text{tr } T^*T < \infty$
 (T is Hilbert-Schmidt)

Let for $A \in \mathfrak{S}_1$ $\text{tr } A = \sum_{n=1}^{\infty} (Ax_n, x_n)$,
 $\{x_n\}$ - orthonormal basis.

Thm 1 $(\mathfrak{S}_1 \subset \mathfrak{S}_2)$

(a) \mathfrak{S}_2 is two-sided *-ideal in $\mathcal{L}(\mathcal{H})$.

(b) If $A, B \in \mathfrak{S}_2$, $A^*B \in \mathfrak{S}_1$
 and

$$(A, B)_2 = \text{tr } A^*B < \infty$$

(c) \mathfrak{S}_2 with the inner product (\cdot, \cdot) is the Hilbert space

$$(d) \|A\|_2^2 = (A, A)_2,$$

$$\|A\| \leq \|A\|_2 \leq \|A\|_1, \quad \|A^*\|_2 = \|A\|_2$$

if $A \in \mathfrak{S}_2$.

(e) Every $A \in \mathfrak{S}_2$ is compact; A compact

$$\in \mathfrak{S}_2 \text{ iff } \sum_{n=1}^{\infty} \mu_n^2(A) < \infty$$

$$\|A\|_2^2 = \sum_{n=1}^{\infty} \mu_n^2(A) < \infty$$

(f) Finite rank operators are dense in \mathfrak{S}_2

(g) $A \in \mathcal{S}_2$ iff $\{\|Ax_n\|\} \in \ell_2$
for some orthonormal basis $\{x_n\}$ for \mathcal{H}

(h) $A \in \mathcal{S}_1$ iff $A = BC$, $B, C \in \mathcal{S}_2$.

Proof

(g) is obvious (follows from properties of \mathcal{S}_1); (g) \Rightarrow (b):

$$|\text{tr } A^* B| = \left| \sum_n (A^* B x_n, x_n) \right|$$

$$\leq \sum_n |(B x_n, A x_n)| \leq \sum_n \|A x_n\| \|B x_n\|$$
$$\leq \left(\sum_n \|A x_n\|^2 \right)^{1/2} \left(\sum_n \|B x_n\|^2 \right)^{1/2}$$

(b) \Rightarrow (a): \mathcal{S}_2 is linear space by (b); ideal - as for \mathcal{S}_1 .

(c) - exercise

(f) - ---

(d) obvious; $\|x\| = 1$

$$(|A|x, |A|x) = \sum_n (|A|x, x_n) (x_n, |A|x)$$
$$\leq \sum_n \| |A|x \| ^2 = \|A\|_2^2;$$

inequality $\|A\|_2 \leq \|A\|_1$ follows from (e):

$$\sum_n \mu_n^2 \leq \left(\sum_n \mu_n \right)^2;$$

(e) follows from Lemma : T^*T is compact $\Rightarrow T$ is compact, formula for the trace: and from

$$\text{tr } |A| = \sum_{n=1}^{\infty} \mu_n(A).$$

Finally, (h) follows from

$$A = U |A|^{1/2} |A|^{1/2} \in \mathcal{S}_1;$$

$$|A|^{1/2}, U |A|^{1/2} \in \mathcal{S}_2.$$

Thm 2 Let $\mathcal{H} = L^2(M, d\mu)$

$A \in \mathcal{S}_2$ iff $\exists K \in L^2(M \times M, d\mu \times d\mu)$ s.t.

$$Af(x) = \int_M K(x,y) f(y) d\mu(y).$$

Moreover,

$$\|A\|_2^2 = \int_M \int_M |K(x,y)|^2 d\mu(x) d\mu(y).$$

Proof See R-S

Finally will study $\mathcal{L}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ - compact operators - closed ideal

Proposition (a) $\text{tr}(aA + bB)$
 $= a \text{tr}A + b \text{tr}B$

(b) $\text{tr}A^* = \overline{\text{tr}A}$

(c) $\text{tr}AB = \text{tr}BA$

$\forall A \in \mathcal{J}_1, B \in \mathcal{L}(\mathcal{H})$.

Proof (a) - (b) obvious. to prove (c), sufficient to consider $B = U$. then

$$\text{tr}AU = \sum_n (AUx_n, x_n)$$

$$= \sum_n (AU \underbrace{U^*}_{I} x_n, U^* x_n)$$

$$= \sum_n (UAx_n, x_n) = \text{tr}UA.$$

Lecture 31

Theorem (a) $\mathcal{J}_1 = \mathcal{K}(\mathcal{H})$

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 $A \mapsto \text{tr}(A \cdot)$

(b) $\mathcal{L}(\mathcal{H}) = \mathcal{J}_1^*$

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$B \mapsto \text{tr}(B \cdot)$.